REFINEMENTS OF SOME EXTREME FORMS

by

C.E. NELSON B.Sc. Hons. (Adel.)

A Thesis submitted for the Degree of

Master of Science

in the University of Adelaide

Department of Mathematics

Dec. 1968

CONTENTS

SUMMARY

Let $f(\underset{\sim}{x}) = f(x_1, x_2, \ldots, x_n)$ be a positive definite quadratic form with determinant D, and let M be the minimum value assumed by f for integral $\underset{\sim}{x} \neq \underset{\sim}{0}$. The relative minimum of f, $\gamma_n(f)$, is defined by

$$\gamma_n(f) = M/D^{1/n} \quad .$$

We let

$$\gamma_n = \underset{f}{\mathrm{MAX}} \; \gamma_n(f)$$

the maximum being taken over all positive definite n-variable forms. We call f extreme if $\gamma_n(f)$ is a local maximum for varying f, and absolutely extreme if $\gamma_n(f)$ is an absolute maximum, so that $\gamma_n(f) = \gamma_n$. Suppose $f(\underset{\sim}{x})$, $g(\underset{\sim}{x})$ are two positive n-variable forms with corresponding lattices Λ, M in E^n. If $\Lambda \subset M$ we say that M refines Λ, and that g refines f.

Recently E.S. Barnes and G.E. Wall published a paper in which they constructed, for each $N=2^n$ $(n=2,3,\ldots)$, a lattice Λ_n in E^N with form f_n which was extreme with

$$\gamma_N(f_n) = (\tfrac{1}{2}N)^{\frac{1}{2}} \quad .$$

The forms f_2, f_3 are absolutely extreme, and for $n \geq 4$, $\gamma_N(f_n)$ exceeds $\gamma_N(f)$ for any other known positive N-variable form f.

This thesis is concerned with the possibility of refining the form f_n to a form with the same minimum M, but with higher relative minimum. This technique has been used by Barnes to construct new classes of extreme forms from known forms. That this method could be applied to f_6 was suggested originally by two papers of J. Leech concerned with packings of the sphere in E^n. By considerably extending this method of refining f_6, I have produced, for each $n \geq 6$, a lattice Δ_n refining Λ_n, and a form g_n refining f_n, with g_n extreme, and

$$\gamma_N(g_n) = N^{2/N}\, 2^{-6/N}\, (\tfrac{1}{2}N)^{\frac{1}{2}} \quad .$$

If n is not too large, this is significantly larger than $\gamma_N(f_n)$, and will improve the lower bound for $\gamma_N (n \geq 6)$. A description of the construction of Δ_n and g_n forms the subject-matter of this thesis.